1/21/26

- HW due Fri
- Math Bunker tutoring has started. In O'Meara Commons (basement of Hayes-Healy) for proof-based classes.

  Sun 6-8

  Mon-Thurs 7-9

- Quiz
  - 10 min
  - fix #3 : Does
  - make sure final answer is marked in each part.

Recap

Def A system of lin eqns is consistent if there's at least one sol'n

A linear system is inconsistent if the RREF includes the row $[0 \ \dots \ 0 \mid 1]$

Note If a lin. system is consistent then it either has a unique solution or infinitely many. (Recall geometry from Monday.)

If it's inconsistent, it has no sol'n.

Def If A is any matrix then the rank of A, denoted $\operatorname{rank}(A)$, is the number of leading 1's in $\operatorname{RREF}(A)$.

Ex
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \operatorname{rank}(A) = 2$

Remark Let A be a matrix of rank 1.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

Assume 1^{st} row isn't 0 (otherwise swap)

Then when you row reduce, all but the 1^{st} row become 0

Conclusion: in a matrix of rank 1, all the rows are scalar mult. of each other.

e.g.
$$\begin{bmatrix} 1 & 3 & 2 & 4 & 0 \\ 2 & 6 & 4 & 8 & 0 \\ 1/2 & 3/2 & 1 & 2 & 0 \end{bmatrix}$$

General Fact:

Thm Consider a lin syst

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + \cdots + a_{2m}x_m &= b_2 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m &= b_n \end{aligned}$$

Call it $\mathcal{L}$.

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$ be the coeff matrix.

Note n rows, m columns for the coefficient matrix, not the augmented matrix.

then

(i) $\text{rank}(A) \leq n$

(ii) $\text{rank}(A) \leq m$

(iii) if $\mathcal{L}$ is inconsistent then $\text{rank}(A) < n$

(iv) if $\mathcal{L}$ has exactly one solution then $\text{rank}(A) = m$

(v) if $\mathcal{L}$ has ∞ly many sol'ns then $\text{rank}(A) < m$

pf/ Let RREF denote the reduced row-echelon form of the <u>coefficient</u> matrix.

(i) $\operatorname{rank}(A) = \#$ leading 1's in RREF $\leq \#$ rows $= n$

(ii) $\operatorname{rank}(A) = \#$ leading 1's in RREF $\leq \#$ vars $= m$

(iii) $\operatorname{rank}(A) = \#$ leading 1's in RREF $< \#$ rows $= n$ since at least one row of the

<u>augmented</u> matrix is $0 \; \cdots \; 0 \mid 1$

(iv) There are no "free" variables so

$\operatorname{rank}(A) = \#$ leading 1's in RREF $= \#$ vars $= m$

(v) Infinitely many solns $\Rightarrow$ at least one "free" variable

$\Rightarrow$ rank $= \#$ leading 1's in RREF $< \#$ vars $= m$ //

<u>Thm</u> Let $\mathcal{L}$ be a lin syst. If $\#$ eqns $< \#$ vars $(n < m)$ then either $\mathcal{L}$ is inconsistent (no soln's) or there are infinitely many solns

pf/ Leave to you. //